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CHARACTERS OF FINITE-DIMENSIONAL PSEUDOREPRESENTATIONS OF GROUPS

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Abstract. The notion of character of a finite-dimensional pseudorepresentation of a group is introduced and the structure of characters of locally bounded pseudorepresentations of connected semisimple Lie groups is described.

§ 1. Introduction

Recall that a mapping π of a given group G into the family of operators on a Banach space E is said to be a *quasirepresentation* of G if

$$\|\pi(g_1g_2)-\pi(g_1)-\pi(g_2)\|\le\varepsilon,\qquad g_1,g_2\in G,$$

for some ε, which is usually assumed to be sufficiently small and is referred to as a *defect* of π and a quasirepresentation of G is said to be a *pseudorepresentation* of G if $\pi(g^n)$ is similar to $\pi(g)^n$, $n\in\mathbb{Z}$, with the help of an operator sufficiently close to the identity operator; cf. [1–4].

§ 2. Generalities concerning characters of finite-dimensional pseudorepresentations of groups

Definition 1. Let π be a finite-dimensional pseudorepresentation of a group G in a space E with basis $e_1,e_2,\dots,e_n$ whose bidual basis in the dual space E^* is $f_1,f_2,\dots,f_n$, $n=\dim E$. Then the function

$$\chi_\pi(g)=\sum_{i=1}^{n}f_i(\pi(g)e_i),\qquad g\in G,$$

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is called the *character* of the pseudorepresentation π.

Remark 1. Obviously, the definition is correct, i.e., the function χ_π does not depend on the choice of a basis $e_1, e_2, \dots, e_n$ in E. This assertion can significantly be sharpened.

Definition 2. Two finite-dimensional pseudorepresentations π and ρ of a group G in spaces E and F are said to be *pointwise equivalent* if there is a family of invertible operators $\theta(g)$, $g \in G$, taking E onto F and such that

$$\theta(g)\pi(g) = \rho(g)\theta(g), \qquad g \in G.$$

Theorem 1. *Let two finite-dimensional pseudorepresentations π and ρ of a group G in spaces E and F be pointwise equivalent. Then the characters χ_π and χ_ρ of the pseudorepresentations π and ρ coincide.*

Proof. The proof is immediate.

Theorem 2. *Let χ_π be the character of a finite-dimensional pseudorepresentations π of a group G in a space E. The restriction of χ_π to every cyclic subgroup H of G is an ordinary character of an ordinary representation of the subgroup H.*

Proof. It follows from the very definition of pseudorepresentation that $\chi_\pi(g^n)$ coincides with the value of the trace on $\pi(g)^n$ for every $g \in G$ and $n \in \mathbb{Z}$, which implies the assertion.

Remark 2. It should be noted that, as in the case of ordinary representations, the coincidence of characters does not imply local equivalence. For example, the mappings

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R},$$

and

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R},$$

give a desired pair of ordinary representations.

§ 3. Main results

The results of [2] and [5] make it possible to describe the characters of locally bounded finite-dimensional finally precontinuous pseudorepresentations of connected locally compact groups. We shall describe here the characters of locally bounded finite-dimensional pseudorepresentations of connected semisimple Lie groups.

Let us recall the definition of Guichardet–Wigner pseudocharacter on semisimple Lie groups.

Definition 3 (see, e.g., [4]**).** Let G be a connected semisimple Lie group and let f be a pseudocharacter on G. Replacing the group G by its universal covering group if necessary (and denoting this group by G again), we may assume that the group G is simply connected (and thus is a product of simply connected simple Lie groups) and treat the pseudocharacter f as a pseudocharacter (denoted by f again) on this simply connected group, or, in other words, as a pseudocharacter on the above direct product of simple Lie groups.

As is known (see, e.g., [4]), this pseudocharacter is a sum of its restrictions to the simple factors.

Every restriction of this kind is either trivial (for instance, if the factor in question is not Hermitian symmetric) or a nonzero multiple of the corresponding Guichardet–Wigner pseudocharacter (see Theorem 3.3.2 of [2]) extended to the entire direct product by zero (if the factor in question is Hermitian symmetric indeed). In what follows, we refer to any linear combination of these extensions of the Guichardet–Wigner pseudocharacters on the Hermitian symmetric factors G_i of G as a *Guichardet–Wigner pseudocharacter* on G.

Theorem 3. *Every character of a finite-dimensional locally bounded pseudorepresentation of a connected semisimple Lie group G can be approximated by a finite linear combination of ordinary characters of ordinary continuous representations of G and, if G has no Hermitian symmetric quotient group, then also of characters of ordinary continuous irreducible unitary representations of the maximal compact quotient group of G and, if G has a nontrivial Hermitian symmetric quotient group, then the linear combination in question includes products of ordinary characters of continuous irreducible unitary representations of the maximal compact quotient group of G by one-dimensional Guichardet–Wigner pseudorepresentations of G (exponentials of some multiples of the Guichardet–Wigner pseudocharacters θ), i.e., by mappings of the form $g \to \exp(ir\theta(g))$, $g \in G$ for some $r \in \mathbb{R}$.*

Proof. The proof follows from Theorem 3.3.14 of [2]. It claims that, for a given pseudorepresentation π, one can construct an approximating mapping of the form

$$\Theta(g)=\begin{pmatrix}\alpha(g) & 0 & 0 & \psi(g)\\ 0 & \beta(g) & 0 & 0\\ 0 & 0 & \Gamma(g) & 0\\ 0 & 0 & 0 & \delta(g)\end{pmatrix},\qquad g\in G,$$

is a pure pseudorepresentation of the group G, and the mapping Θ (or, equivalently, the representation β) is continuous if and only if it is locally bounded. Here α, β, and δ are representations, α and δ are bounded, and *Gamma* is described as follows. If the group G has a nontrivial Hermitian symmetric quotient group, then the mapping γ is a perturbation of the direct sum Γ of (ordinary) products of continuous irreducible unitary representations of the maximal compact quotient group of G by one-dimensional Guichardet–Wigner pseudorepresentations (exponentials of some multiples of the Guichardet–Wigner pseudocharacter θ), i.e., by mappings of the form $g\to\exp(ir\theta(g))$, $g\in G$ for some $r\in\mathbb{R}$. If the group G has no nontrivial Hermitian symmetric quotient groups, then the mapping γ is a perturbation of the direct sum Γ of (ordinary) continuous irreducible unitary representations of the maximal compact quotient group of G. Taking the trace of $\Theta(g)$, $g\in G$, we obtain our result.

§ 4. Question

Is it true that sufficiently norm-close pseudorepresentations have equal characters?

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